The Finite Element Modeling of Static and Stationary Electric and Magnetic Fields

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Abstract– **A finite-element method is described for computing static and stationary electric and magnetic fields directly in terms of the electric or magnetic field strength. In this way the use of (vector) potentials is avoided and a much higher accuracy is achieved with the same computational effort. The curl equation is supplemented with the electromagnetic compatibility relations and a problem is defined which is known to have a unique solution. For satisfying the conditions at the interfaces between different media use is made of both edge and nodal elements. The validity of our approach is illustrated by solving a simple sample problem.**

INTRODUCTION

The finite-element modeling of static and stationary electric and magnetic fields is traditionally carried out in terms of (combinations of) electric and magnetic scalar and vector potentials [1, 2, 3]. These potentials have the advantage that they can be chosen to be continuous across interfaces between different media. In this way difficulties that might arise due to the discontinuity of certain components of the electric (magnetic) field strength (or flux density) across those surfaces are avoided and nodal elements can be used exclusively. The use of potentials, however, has a number of disadvantages, the most important of them being their inefficiency when electric, or magnetic, field strengths or flux densities are required. When field strengths, or fluxes, must be computed, they can only be obtained by carrying out a numerical differentiation on the solution for the potential(s), which causes a large loss of accuracy.

With the advent of edge and face elements it has become possible to solve the difficulties due to discontinuities in medium properties and, consequently, in certain components of the electric and/or magnetic field strength (or flux density), at the element level. Because of this it has become possible to formulate static and stationary electric and magnetic field problems directly in terms of the electric (or magnetic) field strength by using edge elements which are known to allow jumps in normal components of the field across an interface. Alternatively it is possible to formulate the problem in terms of the electric (or magnetic) flux density by using face elements which are known to allow jumps in the tangential components of the flux density across an interface. In the present paper we shall discuss the formulation of static and stationary field problems directly in terms of the electric (or magnetic) field strength. A formulation of field problems in terms of the electric (or magnetic) flux density can be obtained in the same way by replacing edge elements by face elements. Face elements, however, are less efficient than edge elements (more expansion functions are required for the same domain of computation) and because of this the formulation using edge elements together with nodal elements is preferable.

In our paper we shall describe the method for the case of stationary magnetic fields, the method for computing static electric fields runs along the same lines. For obtaining optimal computational efficiency we use both edge and nodal elements [4, 5], using nodal elements in the homogeneous subdomains of the configuration and using edge elements along the interfaces between those homogeneous subdomains and near singularities.

FIELD EQUATIONS

Let the domain $\mathcal{D} \subset \mathbf{R}^3$ be a closed, connected domain of computation (see Fig. 1) with outer boundary $\partial\mathcal{D} = \partial\mathcal{D}_{\mathrm{E}} \cup \partial\mathcal{D}_{\mathrm{H}}$ (with $\partial\mathcal{D}_{\mathrm{E}} \cap \partial\mathcal{D}_{\mathrm{H}} = \emptyset$). The materials in the domain of computation may be inhomogeneous and anisotropic. Let $\mathcal{I}$ denote the interfaces between adjacent subdomains of $\mathcal{D}$ with different medium parameters, and let $\boldsymbol{\nu}$ denote the unit vector along the normal to either an interface or the outer boundary $\partial\mathcal{D}$. The volume source density of the imposed electric current $\boldsymbol{J}^{\mathrm{imp}}$ is assumed to be known throughout $\mathcal{D}$ as a function of the position vector $\boldsymbol{r}$. The values of the boundary conditions $\boldsymbol{\nu} \times \boldsymbol{E}^{\mathrm{ext}}$ and $\boldsymbol{\nu} \times \boldsymbol{H}^{\mathrm{ext}}$ are known as a function of $\boldsymbol{r}$ at $\partial\mathcal{D}_{\mathrm{E}}$ and $\partial\mathcal{D}_{\mathrm{H}}$, respectively.

Assuming known medium parameters and a known initial field distribution this defines in the time domain a problem with a unique solution [6].

In the stationary limit of the time-domain electromagnetic field equations, the equations that apply to the magnetic field strength reduce to the well known curl equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{\mathrm{imp}}. \tag{1}$$

This equation is supplemented with the condition

$$\boldsymbol{\nu} \times \boldsymbol{H} \text{ continuous across sourcefree interfaces} \tag{2}$$

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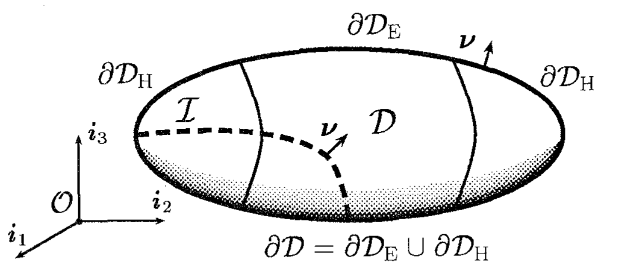

Fig. 1. The domain of computation $\mathcal{D}$.

and the boundary condition

$$\boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{\nu} \times \boldsymbol{H}^{\rm ext} \text{ on } \partial\mathcal{D}_{\rm H}. \tag{3}$$

Note that the value $\boldsymbol{\nu} \times \boldsymbol{E}^{\rm ext}$ which is prescribed on $\partial\mathcal{D}_{\rm E}$ can, because of the decoupling between the electric and the magnetic field, not be used as a boundary condition.

The stationary limit of the time-domain electromagnetic field equations, (1) to (3) does not define a problem with a unique solution. For obtaining a formulation with a unique solution these equations must be supplemented with the compatibility relations.

Compatibility Relations

Compatibility relations are properties of the field that are direct consequences of the field equations and that must be satisfied to allow them to have a solution [7].

The local compatibility relations are obtained as the static versions of the electromagnetic compatibility relations for transient fields [8, 9].

For stationary magnetic fields the volume divergence relation reduces to

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4}$$

where $\boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H}$.

For stationary magnetic fields the surface divergence relation reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{B} \quad \text{continuous across } \mathcal{I}. \tag{5}$$

For stationary magnetic fields the continuity of the normal flux at the outer boundary yields

$$\boldsymbol{\nu} \cdot \boldsymbol{B} = \boldsymbol{\nu} \cdot \boldsymbol{B}^{\rm ext} \text{ on } \partial\mathcal{D}_{\rm E}, \tag{6}$$

where $\boldsymbol{\nu} \cdot \boldsymbol{B}^{\rm ext}$ denotes the normal component of the external, known, magnetic flux density. This relation is used to replace the boundary condition on $\partial\mathcal{D}_{\rm E}$ for stationary magnetic fields.

Equations (1) to (3), together with the compatibility relations (4) to (6) define a problem which can be shown to have a unique solution.

In case the surface $\partial\mathcal{D}_{\rm H}$ consists N disjoint, connected subsurfaces, (see Fig. 1) $\boldsymbol{H}$ has to satisfy N-1 additional global compatibility relations of the type

$$\int_{\Gamma_k} \boldsymbol{H} \cdot \boldsymbol{\tau} \, {\rm d}s = U^{\rm ext}_{{\rm m},\,k}, \qquad k = 2, ..., {\rm N}, \tag{7}$$

where Γ_k are N-1 piecewise smooth curves that join one prefered subsurface with each of the other subsurfaces, $\boldsymbol{\tau}$ is the unit vector along the tangent to Γ_k and $U^{\rm ext}_{{\rm m},\,k}$ are the impressed (known) values of the magnetic potential difference along Γ_k.

Implementation Aspects

Equations (1) to (6) are implemented in a finite-element code, replacing the continuous variables by their discretized counterparts, employing both linear edge and linear nodal elements. In homogeneous subdomains nodal elements are used. Edge elements are used along interfaces between different media, along (parts of) the outer bondary [4, 5] that are not parallel to a Cartesian plane and near reentrant corners in the outer boundary. By using edge elements near these "singularities" the large errors made by using nodal elements at those locations are avoided and a simpler, direct, method for prescribing the tangential field components along the outer boundary is obtained. The resulting expansion can be written as

$$\boldsymbol{H} = \sum_i h_i \boldsymbol{W}_i^{({\rm E,N})}, \tag{8}$$

where $\boldsymbol{W}_i$ denotes the edge (E) or nodal (N) expansion used [4]. Note that, because of employing consistently linear elements a local approximation error $O(h^2)$, where h denotes the length of the longest edge of a tetrahedron used locally, is obtained in the representation of the magnetic field strength $\boldsymbol{H}$.

A system of algebraic equations for the expansion coefficients h_i is obtained in a standard manner by using weighted residuals. The curl equation (1) is multiplied with the set of appropriate vectorial weighting functions

$$\boldsymbol{T}_j = \nabla \times \boldsymbol{W}_j^{({\rm E,N})}, \quad \forall j. \tag{9}$$

The divergence equation (4) is multiplied with the set of scalar weights

$$t_j = 1/||\boldsymbol{\mu}||^2 \; \nabla \cdot (\boldsymbol{\mu} \cdot \boldsymbol{W}_j^{({\rm E,N})}), \quad \forall j, \tag{10}$$

where the norm is taken such that reduces to μ for isotropic media. Integrations are carried out over the subdomains in which both the relevant weighting function and the relevant expansion function are nonzero

$$\sum_i h_i \int_{\mathcal{D}_{i,j}} \boldsymbol{T}_j \cdot \left(\nabla \times \boldsymbol{W}_i^{({\rm E,N})} \right) = \int_{\mathcal{D}_j} \boldsymbol{T}_j \cdot \boldsymbol{J}^{\rm imp}, \quad \forall j, \tag{11}$$

$$\sum_i h_i \int_{\mathcal{D}_{i,j}} t_j \left(\nabla \cdot (\mu\, \boldsymbol{W}_i^{(\mathrm{E,N})})\right) = 0, \quad \forall j, \tag{12}$$

where $\mathcal{D}_{i,j} = \mathcal{D}_i \cap \mathcal{D}_j$ is the cross-section of the span $\mathcal{D}_j$ of the weighting function $\boldsymbol{T}_j$ and the span $\mathcal{D}_i$ of the expansion function $\boldsymbol{W}_i$. The set of Eqs. (11) and (12) satisfy the requirements for a consistent Galerkin method with both the weighting and the weighted functions being the curl and the divergence of the expansion functions. The choice for the weighting functions $\boldsymbol{T}_i$ and t_i yields Eqs. (11) and (12) to be dimensionally uniform. The two equations can be thus summed which method is commonly used for obtaining a square system of linear equations. The final form of the weighted residuals equations is

$$\sum_i h_i \left[\int_{\mathcal{D}_{i,j}} \boldsymbol{T}_j \cdot \left(\nabla \times \boldsymbol{W}_i^{(\mathrm{E,N})}\right) + \int_{\mathcal{D}_{i,j}} t_j \left(\nabla \cdot (\mu\, \boldsymbol{W}_i^{(\mathrm{E,N})})\right) \right] = \int_{\mathcal{D}_j} \boldsymbol{T}_j \cdot \boldsymbol{J}^{\mathrm{imp}}, \quad \forall j. \tag{13}$$

Note that the condition on the curl and the divergence in each separate tetrahedron is accounted for in a number of different equations of the type (13).

The boundary condition (3) and the local compatibility relations (5) and (6) are subsequently imposed in a strong way by imposing the implied relations between the expansion coefficients. These relations replace some of the equations of the type (13), without causing the conditions on the curl and divergence on the relevant equation to be lost because they still appear in other equations. The actual equations to be replaced are chosen such that the relevant condition yields an equation with the maximum absolute value of its coefficients located on the diagonal of the system of linear equations. In this way the condition of the system of equations is optimized.

EXAMPLE

For demonstrating the validity of our approach we shall give some results for a simple single slot problem. The configuration is depicted in Fig. 2. For discretizing the domain of computation $\mathcal{D}$ $(0 < x < 2,\ 0 < y < 4,\ 0 < z < 0.3)$, it was subdivided in a non-uniform mesh consisting of $N_x * N_y * N_z = 14 * 28 * 3$ bricks, each brick being subdivided into six tetrahedra.

The field is excited by an imposed current distribution with a current density of $\boldsymbol{J}^{\mathrm{imp}} = \mathrm{J}_0 \boldsymbol{i}_z$, $\mathrm{J}_0 = 1\mathrm{A/m}^2$ in the region $0 \le x \le 0.9$, $1.1 \le y \le 2$, $-\infty \le z \le \infty$, and zero elsewhere. The iron is assumed to be isotropic, having a relative permeability $\mu_{\mathrm{r}} = 1000$, the relative permeability of the surrounding material was set to 1. The following boundary conditions were imposed: at the planes z=0, z=0.3 and x=2 the normal component of the magnetic flux density was set to zero (see (6)), at the remaining

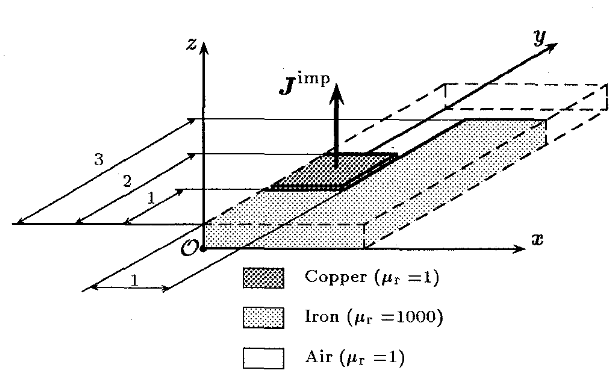

Fig. 2. The configuration

parts of the outer boundary the tangential components of the magnetic field strength were set to zero (see (3)). With these choices the total number of unknowns amounted to $N_{\mathrm{unk}} = 6236$, $N_{\mathrm{bnd}} = 476$ of them being specified explicitly using (3). The system of equations was first solved by using a direct method of solution which choice was made to prove that the matrix is non-singular. A subsequent solution of the same system of equations using an iterative solver with ICCG preconditioning yielded the same result, while using much less computation time and storage space.

Although being unrealistic as an object one might find in engineering, this problem is, on the one hand, simple enough to allow the accuracy of its solution to be easily judged, even by inspection, and on the other hand, complicated enough for not being trivial.

Observing the results for the x-component as depicted in Fig. 3, one notices the expected "linear" behaviour of the field as a function of y in the region $1 \le y \le 2$ and the singularity at the top edge of the iron region. As expected, the field in iron turns out to be almost equal to zero. A slight deviation from this is observed near the singularity mentioned earlier. The results for the y−component are depicted in Fig. 4, note the change in the angle of observation. As regards the z-component of the computed magnetic field strength, which should theoretically be identical to zero, we note that it deviated from zero only in the immediate vicinity of the iron-air interface, having a maximum value of only a few % of the local x- or y-component. Note that this is not unexpected since the interface is the type of "singularity" along which edge elements are used and since edge elements use non-Cartesian bases for representing the field strength.

As regards the contrast in the permeability between iron and air it was observed that, away from locations where the field has singularities, a contrast of a factor 1000 was represented well in the results for the normal components of the magnetic field strength, which should reflect this contrast. In the immediate vicinity of a singularity the continuity of the normal flux was satisfied less accurately. Higher contrasts were represented less accurately, possibly due to loss of accuracy in the solution of the system matrix.

DISCUSSION

The present finite-element formulation of stationary magnetic field problems yields highly accurate results with a minimum of computational effort. The formulation used requires that a number of different conditions (the curl equation, the compatibility relations and the boundary conditions) must be implemented, e.g. by "adding" them to the system matrix. For most of these conditions a wide range of options for implementing them is available. In the vicinity of locations where the field is expected to show singularities it is also necessary to choose the meshing carefully so as to obtain maximum benefits from the use of edge elements. Research into these aspects of our method is still in progress.

It was already mentioned above that our opproach yield a local approximation error $O(h^2)$ in the representation of the magnetic field strength. Because of additional errors along the interfaces and, in particular near the edges in the configuration where the field strength, or derivatives of it, may be singular, it is not possible to estimate the global error theoretically. Experimentally, however, we have found promising results for convergence as a function of decreasing mesh size. Note that a method based on the use of (vector) potentials would yield a local approximation error $O(h)$ in the representation of the magnetic field strength for the same computational effort.

CONCLUSIONS

We have presented a new formulation for the computation of stationary magnetic fields by computing this field directly in terms of the magnetic field strength itself. In this way we obtain a highly efficient method that is superior to existing formulations using potentials because of a much higher accuracy with the same computational effort. Numerical examples demonstrating the validity, the accuracy and the efficiency of our method was presented.

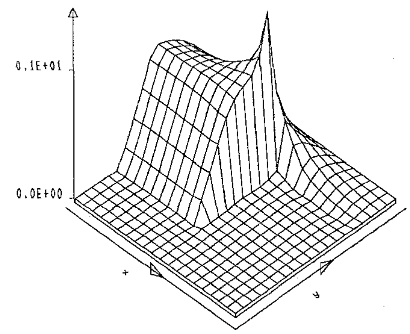

Fig. 3. H_x in the plane $z = 0.155$, $x \in [0,2]$, $y \in [0,4]$. $H_{x\ \mathrm{minimum}} = -0.0332$ A/m, $H_{x\ \mathrm{maximum}} = 1.23$ A/m.

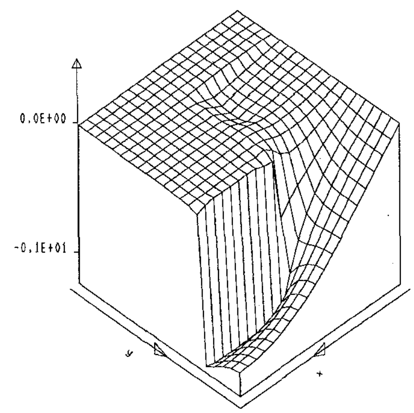

Fig. 4. H_y in the plane $z = 0.155$, $x \in [0,2]$, $y \in [0,4]$. $H_{y\ \mathrm{minimum}} = -1.24$ A/m, $H_{y\ \mathrm{maximum}} = 0.0898$ A/m.

REFERENCES

[1] N. Ida and J. P. A. Bastos, *Electromagnetics and calculation of fields.* Springer-Verlag Inc., New York, 1992.

[2] J. Jin, *The finite element method in electromagnetics.* John Wiley & sons, Inc., New York, 1993.

[3] P. P. Silvester and R. L. Ferrari, *Finite elements for electrical engineers.* (second edition) Cambridge University Press, Cambridge, 1989.

[4] G. Mur, "Optimum Choice of Finite Elements for Computing Three-Dimensional Electromagnetic Fields in Inhomogeneous Media," in *IEEE Transactions on Magnetics*, vol. 24, no. 1, pp. 330-333, January 1988.

[5] G. Mur, "Edge elements, their advantages and their disadvantages," in *IEEE Transactions on Magnetics*, vol. 30, no. 5, pp. 3552-3557, September 1994.

[6] J. A. Stratton, *Electromagnetic Theory.* McGraw-Hill Book Company, New York, 1941, p. 486.

[7] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity,* The Syndics of the Cambridge University Press, London, 1959, p. 49.

[8] G. Mur, "Compatibility Relations and the Finite-Element Formulations of Electromagnetic Field Problems," in *IEEE Transactions on Magnetics*, vol. 30, no. 5, pp. 2972-2975, September 1994.

[9] I. E. Lager and G. Mur, "Compatibility Relations for Time-Domain and Static Electromagnetic Field Problems," in *Applied Computational Electromagnetics Society Journal*, vol. 9, no. 2, pp. 25-29, July 1994.