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On the Causes of Spurious Solutions in Electromagnetics

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The problem of spurious solutions in finite-element methods for electromagnetics is discussed with the explicit aim to re-open the discussion on this much debated subject. First, an overview is given of existing ideas about the causes of spurious solutions together with the solutions that are proposed. Subsequently it is concluded that the cause of spurious solutions may differ from what is usually suggested or claimed, and an alternative cause for spurious solutions is proposed. Finally, suggestions are given about how a finite-element method should be designed that is free of spurious solutions.

Keywords finite-element method, spurious solutions, static electromagnetic fields

Introduction

Many finite-element methods for static or dynamic fields in electromagnetics suffer from spurious solutions. Spurious solutions are found in both driven and eigenvalue problems; in the latter case they are referred to as spurious modes. Spurious solutions can usually be recognized by a violation of the divergence equations, but the curl equations may also be violated. Over the years various explanations and appertaining solutions for this ever recurring problem have been proposed, of which we mention the following discussions.

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Nodal elements. Many authors relate spurious solutions to the use of nodal elements and claim to be able to solve this difficulty by using a special type of elements, the so-called edge elements that are sometimes also referred to as vector finite elements.

As regards *nodal elements* we note that nodal elments are nothing but piecewise linear (or higher order) functions, and by using nodal elements for representing an unknown (vector) field, one obtains the most general piecewise linear (or higher order) solution space possible. Nodal elements neither favor nor disfavor any point in this solution space; it is the finite-element formulation used that should find the location in the solution space that we call the best possible solution (in that solution space) for the set of equations to be solved. It is not clear how nodal elements could be blamed for the failure of finding this best possible solution in the solution space they provide.

As regards the use of *edge elements*, we note that it has been known for some time now that edge elements do not eliminate spurious solutions by themselves. Edge elements restrict the solution space available in the sense that *inside* each element the solution is free of divergence. However, they still allow spurious solutions. An explicit example of this was given by Mur (1994b, 1998), and in this respect they are neither better nor worse than nodal elements.

Differential forms. Bossavit (1991, 1998) claimed that only the use of a differential forms formulation can remedy the problem. A disadvantage of his approach seems to be that it involves a very high degree of mathematical sophistication that follows from the *geometrical* interpretation given to the electromagnetic field (Baldomir & Hammond, 1996). In this respect, it is worthwhile mentioning the difficulty to model the Hodge operator (Bossavit & Kettunen, 1999), whose efficient implementation in a computer code is still a subject of research. Under these circumstances, it is understandable that a commercial implementation of this approach is not yet available.

The div-curl method. Again others state that spurious solutions are avoided by using the div-curl method (Jiang, Wu, & Povinelli, 1996). We do agree that both the divergence and the curl should be made part of the finite-element formulation of an electromagnetic field problem (Mur, 1994a) but we have found, and will show, examples of spurious solutions when using this formulation.

Domain-integrated field equations. Finally, we mention the recently developed domain-integrated field equation method (de Hoop & Lager, 1998, 1999) which is of the finite-element type and which is free of spurious solutions. Unfortunately, the general application of this innovating method is blocked by its extreme inefficiency which, for the time being, makes it of academic interest only.

In our discussion of the subject of spurious solutions we will, for simplicity and clarity, confine ourselves to the case of computing static magnetic fields directly in terms of the magnetic field strength. First, a variational formulation of the static magnetic field problem will be given. Subsequently an analysis will be made of possible causes of spurious solutions when using this formulation.

Although it cannot be claimed that the results of the analysis of a method for static fields to be presented are automatically applicable to time-harmonic and/or transient electromagnetic fields using similar formulations, we surmise that the analysis yields sufficient new insight to make it of interest to people engaged in the design of finite-element methods for nonstatic or even high-frequency electromagnetic fields. Finally we will discuss new options for formulating finite-element methods for (electro)magnetic field computation that are free of spurious solutions.

A Static Field Formulation for the Magnetic Field

The formulation to be used is based on a local vectorial model of the field in which the vector functions $\boldsymbol{H}$ and $\boldsymbol{B}$ are both taken to continuously depend on the position vector and to be pointwise related through a constitutive equation. In this manner, there is but one single vectorial field quantity to be computed, which makes this choice the most economical one, from a computational point of view.

This choice for modeling the field quantities results in a *local* (understood in the sense of "pointwise") description of the field. As a consequence, the mathematical model will necessarily be a *differential* one, implying that the differentiability of the modeled field quantities should be ensured throughout the domain of computation $\mathcal{D}$ except, perhaps, on some sets of measure zero in R^3. On such sets, the differential operators need to be replaced by appropriate conditions on the components of the field quantities. Since this replacement is not always applicable (for example, at points where the *geometrical singularities* of the domain of computation induce possible singular behaviors of the modeled field quantities), certain (simplifying) choices have to be made with respect to the modeling of the field quantities in such regions (these choices being discussed later).

The Field Equations

For static magnetic fields, the following equations apply (Lager & Mur, 1998b):

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}^{\text{imp}} \quad \text{inside } \mathcal{D}\cap\mathcal{I}, \tag{1}$$

$$\nabla\cdot\boldsymbol{B} = 0 \quad \text{inside } \mathcal{D}\cap\mathcal{I}, \tag{2}$$

$$\boldsymbol{B} = \mu\,\boldsymbol{H} \quad \text{inside } \mathcal{D}\cap\mathcal{I}, \tag{3}$$

(1) entailing the *local* compatibility relation

$$\nabla\cdot\boldsymbol{J}^{\text{imp}} = 0, \quad \text{inside } \mathcal{D}\cap\mathcal{I}. \tag{4}$$

Equations (1)–(3) are supplemented by the interface conditions:

$$\nu\times\boldsymbol{H} = \text{continuous across interfaces}, \tag{5}$$

$$\nu\cdot\boldsymbol{B} = \text{continuous across interfaces}, \tag{6}$$

and by the two possible types of boundary conditions:

$$\boldsymbol{n}\times\boldsymbol{H} = \boldsymbol{n}\times\boldsymbol{H}^{\text{ext}} \quad \text{on } \partial\mathcal{D}_{\boldsymbol{H}}, \tag{7}$$

$$\boldsymbol{n}\cdot\boldsymbol{B} = \boldsymbol{n}\cdot\boldsymbol{B}^{\text{ext}} \quad \text{on } \partial\mathcal{D}_{\boldsymbol{B}}. \tag{8}$$

Equations (1)–(3) and (5)–(8), subject to the compatibility relation (4), define a problem with a unique solution. Note that this is also *the only* set of equations pertaining to the stationary magnetic field for which the uniqueness of the solution can be rigorously proven.

The Discretized Field Equations

Based on the mathematical model of the field, a computational model can be derived if the following requirements with respect to the discretization of the field quantities can be achieved:

1. The expansion functions should ensure the continuity of the discretized counterpart $\tilde{\boldsymbol{H}}$ of the magnetic field strength $\boldsymbol{H}$ inside interface-free subdomains.
2. The expansion functions should *explicitly* satisfy the interface conditions (5) and (6) and zero (homogeneous) boundary conditions of the type (7) and (8) (with the constitutive equation (3) being substituted in (6) and (8)).

It has been shown by Lager and Mur (1998b) that these requirements can be met by employing *generalized Cartesian expansion functions* (Lager & Mur, 1989a) defined on a tetrahedral mesh.

For readers not yet familiar with generalized Cartesian expansion functions, we note here that (a) they are identical to the classical nodal expansion functions in homogeneous regions and that (b) they have been designed such that exact interface conditions (5) and (6) are included in them along locally plane interfaces between regions with different material properties, while retaining all other properties of nodal expansion functions. It should be mentioned that generalized Cartesian expansion functions cannot be used at points and along edges where the modeled field quantities may have singularities. In such cases the use of generalized Cartesian expansion functions associated with incomplete simplicial stars or expansion functions that account for the relevant type of singularity was proposed.

The following methodology for numerically solving the system of equations (1)–(3) and (5)–(8) was proposed by Lager and Mur (1998b):

1. *Constructing* a vectorial function $\boldsymbol{H}_{\partial\mathcal{D}}$ that accounts for nonzero boundary conditions by using the expressions

$$\boldsymbol{n}\times\boldsymbol{H}_{\partial\mathcal{D}} = \boldsymbol{n}\times\boldsymbol{H}^{\text{ext}} \quad \text{on } \partial\mathcal{D}_{\boldsymbol{H}}, \tag{9}$$

$$\boldsymbol{n}\cdot(\mu\,\boldsymbol{H}_{\partial\mathcal{D}}) = \boldsymbol{n}\cdot\boldsymbol{B}^{\text{ext}} \quad \text{on } \partial\mathcal{D}_{\boldsymbol{B}}. \tag{10}$$

2. *Computing* a vectorial function $\boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}}$ by solving the equations

$$\nabla\times\boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}} = \boldsymbol{J}^{\text{imp}} - \nabla\times\boldsymbol{H}_{\partial\mathcal{D}} = \boldsymbol{J}^{\text{eqv}} \quad \text{inside } \mathcal{D}\cap\mathcal{I}, \tag{11}$$

$$\nabla\cdot\left(\mu\,\boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}}\right) = -\nabla\cdot(\mu\,\boldsymbol{H}_{\partial\mathcal{D}}) = \rho_{\text{m}}^{\text{eqv}} \quad \text{inside } \mathcal{D}\cap\mathcal{I}, \tag{12}$$

   with zero boundary conditions and source-free interfaces. Note that the effect of nonzero boundary conditions is now taken into account in $\boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}}$ by means of the equivalent *volume* source distributions $\boldsymbol{J}^{\text{eqv}}$ and $\rho_{\text{m}}^{\text{eqv}}$ (the latter acting as an "impressed" volume distribution of fictitious magnetic charge).
3. *Computing* $\boldsymbol{H}$ by using the expression

$$\boldsymbol{H} = \boldsymbol{H}_{\partial\mathcal{D}} + \boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}}. \tag{13}$$

The "exact" magnetic field strength $\boldsymbol{H}$ is now replaced by its discretized counterpart $\tilde{\boldsymbol{H}}$ (and, correspondingly, the constituents $\boldsymbol{H}_{\partial\mathcal{D}}$ and $\boldsymbol{H}_{\mathcal{D}\cap\mathcal{I}}$ by $\tilde{\boldsymbol{H}}_{\partial\mathcal{D}}$ and $\tilde{\boldsymbol{H}}_{\mathcal{D}\cap\mathcal{I}}$, respectively). In view of the continuity requirements, generalized Cartesian expansion functions are employed, exclusively (the superscript $\mathcal{C}$ is employed to indicate the fact that we use generalized Cartesian expansion functions). The discretized (approximate) field quantities $\tilde{\boldsymbol{H}}$, $\tilde{\boldsymbol{H}}_{\partial\mathcal{D}}$, and $\tilde{\boldsymbol{H}}_{\mathcal{D}\cap\mathcal{I}}$ can be written as

$$\tilde{\boldsymbol{H}} = \sum_{I} H_I\, \boldsymbol{W}_I^{\mathcal{C}} \quad \text{for } \boldsymbol{W}_I^{\mathcal{C}} \in \mathrm{W}_{\mathcal{D}}, \tag{14}$$

$$\tilde{\boldsymbol{H}}_{\partial\mathcal{D}} = \sum_{I} H_I\, \boldsymbol{W}_I^{\mathcal{C}} \quad \text{for } \boldsymbol{W}_I^{\mathcal{C}} \in \mathrm{W}_{\partial\mathcal{D}}, \tag{15}$$

$$\tilde{\boldsymbol{H}}_{\mathcal{D}\cap\mathcal{I}} = \sum_{I} H_I\, \boldsymbol{W}_I^{\mathcal{C}} \quad \text{for } \boldsymbol{W}_I^{\mathcal{C}} \in \mathrm{W}_{\mathcal{D}\cap\mathcal{I}}, \tag{16}$$

where $\mathrm{W}_{\mathcal{D}}$, $\mathrm{W}_{\partial\mathcal{D}}$, and $\mathrm{W}_{\mathcal{D}\cap\mathcal{I}}$ denote the complete sets of expansion functions required for generating $\tilde{\boldsymbol{H}}$, $\tilde{\boldsymbol{H}}_{\partial\mathcal{D}}$, and $\tilde{\boldsymbol{H}}_{\mathcal{D}\cap\mathcal{I}}$, respectively. Evidently, $\mathrm{W}_{\mathcal{D}} = \mathrm{W}_{\partial\mathcal{D}} \oplus \mathrm{W}_{\mathcal{D}\cap\mathcal{I}}$.

Due to the convenient choice of the sets of expansion functions, constructing the constituent $\tilde{\boldsymbol{H}}_{\partial\mathcal{D}}$ is elementary (Lager & Mur, 1998b). Further, it can be easily shown that, for domains of computation where *no geometrical singularities may be present*, the expansion functions in $\mathrm{W}_{\mathcal{D}\cap\mathcal{I}}$ automatically satisfy the continuity conditions across interfaces and zero boundary conditions. The vector space $\mathrm{X}_{\mathcal{D}\cap\mathcal{I}}$ spanned by the set of expansion functions $\mathrm{W}_{\mathcal{D}\cap\mathcal{I}}$ represents an adequate environment for formulating the solution of (11) and (12) as a variational problem. The solution of these equations then translates into the solution of the system of linear, algebraic equations

$$\begin{aligned}
&\sum_{I} H_I \left(\sum_{\mathcal{T}_u \in (\mathcal{S}_I \cap \mathcal{S}_J)} \int_{\mathcal{T}_u} \left\{ \left(\nabla\times \boldsymbol{W}_J^{\mathcal{C}}\right)\cdot \left(\nabla\times \boldsymbol{W}_I^{\mathcal{C}}\right) + \frac{1}{\|\mu\|_{\mathcal{T}_u}^2} \left[\nabla\cdot \left(\mu\, \boldsymbol{W}_J^{\mathcal{C}}\right)\right] \left[\nabla\cdot \left(\mu\, \boldsymbol{W}_I^{\mathcal{C}}\right)\right] \right\} \mathrm{d}V \right) \\
&\quad = \sum_{\mathcal{T}_u \in \mathcal{S}_J} \left(\int_{\mathcal{T}_u} \left\{ \left(\nabla\times \boldsymbol{W}_J^{\mathcal{C}}\right)\cdot \boldsymbol{J}^{\mathrm{eqv}} + \frac{1}{\|\mu\|_{\mathcal{T}_u}^2} \left[\nabla\cdot \left(\mu\, \boldsymbol{W}_I^{\mathcal{C}}\right)\right] \rho_{\mathrm{m}}^{\mathrm{eqv}} \right\} \mathrm{d}V \right) \\
&\qquad \forall\ \boldsymbol{W}_J^{\mathcal{C}} \in \mathrm{W}_{\mathcal{D}\cap\mathcal{I}},
\end{aligned} \tag{17}$$

where (a) $\mathcal{S}_I$ and $\mathcal{S}_J$ denote the simplicial stars of the nodes to which the expansion functions $\boldsymbol{W}_I^{\mathcal{C}}$ and $\boldsymbol{W}_J^{\mathcal{C}}$, respectively, pertain; and (b) $\|\mu\|_{\mathcal{T}_u}$ is taken as the maximum value of the permeability inside the tetrahedron $\mathcal{T}_u$.

By solving the system (17), the constituent $\tilde{\boldsymbol{H}}_{\mathcal{D}\cap\mathcal{I}}$ can be computed, after which the discretized (approximate) counterpart $\tilde{\boldsymbol{H}}$ of $\boldsymbol{H}$ is fully determined.

Exceptional Conditions

Lager and Mur (1998a) have shown that generalized Cartesian expansion functions cannot be employed along lines and at points located on interfaces and/or on the outer boundary where the normal is not uniquely defined. In order to extend the applicability of the local vectorial model to configurations that contain this type of geometrical singularities, the use of generalized Cartesian expansion functions associated with incomplete simplicial stars was proposed by Lager and Mur (1998b).

The manner in which expansion functions associated with incomplete simplicial stars are defined is aimed at introducing *exactly* as many degrees of freedom in the local expansion as are needed for being able to cope with the local geometrical discontinuities. The price to pay is the introduction of additional interfaces (that were not present in the original configuration) where the local representation changes abruptly. For being able to adequately model the field quantities "at" interfaces and/or "at" the outer boundary, the boundaries of the incomplete simplicial stars are chosen to be always located *inside interface-free regions*.

An adverse effect of the use of expansion functions associated with incomplete simplicial stars is caused by the abrupt change of the local representation of $\boldsymbol{H}$ at the boundaries of the incomplete simplicial stars. As a consequence, unphysical surface distributions of electric surface currents and of (fictitious) magnetic surface charges are generated along the boundary between the two incomplete simplicial stars. The effect of these unphysical surface distributions can be eliminated by applying the following methodology:

- computing the total supplementary electric current and/or fictitious magnetic charge on each face located on the boundary of an incomplete simplicial star,
- deducting the relevant surface currents and/or fictitious magnetic charges (possibly converted into equivalent *volume* currents and/or distributions of fictitious magnetic charges).

As for the use of expansion functions that account for the relevant type of singularity, we note that employing them requires an a priori knowledge of the type of singularity to be modeled. Since this only applies to some very few, particular cases, it then follows that this strategy cannot be considered as a feasible option in a generally applicable code.

The Resulting System of Linear, Algebraic Equations

From (17) it is obvious that on a specified *row* in the resulting system of linear, algebraic equations (corresponding to a specified value of the index J), there are nonzero entries that correspond to expansion coefficients associated with the nodes pertaining to the simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$ only.

Assume now that the simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$ is entirely located *inside* a homogeneous subdomain of the domain of computation $\mathcal{D}$, characterized by the constant permeability μ. In this case, based on the choice for expansion function described in Lager and Mur (1998b), standard Cartesian expansion functions are employed, exclusively. Row J in the system of linear equations then reads

$$\sum_{I\in\mathcal{S}_J} H_I \left\{ \sum_{\mathcal{T}_u\in\mathcal{S}_I} \int_{\mathcal{T}_u} \left[\left(\nabla\times \boldsymbol{W}_J^{\mathcal{C}}\right)\cdot\left(\nabla\times \boldsymbol{W}_I^{\mathcal{C}}\right)+\left(\nabla\cdot \boldsymbol{W}_J^{\mathcal{C}}\right)\left(\nabla\cdot \boldsymbol{W}_I^{\mathcal{C}}\right)\right]\mathrm{d}V\right\}$$
$$= \langle \text{source terms} \rangle, \tag{18}$$

where the conventional notation $I \in \mathcal{S}_J$ is employed for denoting a summation that is carried out over the indices I of the expansion functions associated with the nodes that belong to the simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$.

We now want to investigate to what extent the system of linear equations of the type (18) really models the problem to be solved, as described by (1)–(3). To that aim we will examine the local properties of our model and consider a number of different typical situations.

Internal Simplicial Star. Let us first study the simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$ of node $\mathcal{N}_{n(J)}$ that is entirely located *inside* a homogeneous subdomian of the domain of the computation $\mathcal{D}$ and that is characterized by the constant permeability μ. The solution of (1)–(3) in this simplicial star is governed by the three equations of the type (18) originating from this simplicial star. As regards the constitutive equation (3), there is no concern about its satisfaction; it is used for the elimination of $\boldsymbol{B}$ and consequently it is satisfied exactly throughout the simplicial star. What remains to be accounted for is the satisfaction of the conditions on the curl (1) and on the divergence (2). Note that these equations together constitute four independent conditions to be imposed on the field strength $\tilde{\boldsymbol{H}}$ and that we only have three independent equations of the type (18) for imposing these four conditions. Moreover, $\tilde{\boldsymbol{H}}$ offers only three independent degrees of freedom for accommodating these four conditions over the simplicial star. It can be argued here that the variational formulation (17) used provides the "best possible solution" for this combination of conditions.

It is also noteworthy to mention that there is a more subtle flaw in the reasoning above. Inside homogeneous subdomains, the norm that was at the origin of the minimization procedure combines the error in modeling the curl of $\tilde{\boldsymbol{H}}$ and the divergence *of the same quantity*. Now, following a reasoning based on the trace theorem (Choquet-Bruhat & DeWitt-Morette, 1982, p. 506), it can be asserted that the curl is associated with a vectorial quantity whose *tangential components* are to be continuous at a surface of discontinuity (and any common face of adjacent tetrahedra *is* such a surface, at least for the derivatives of the expansion functions) while the divergence is associated with a vectorial quantity whose *normal component* is to be continuous at such a surface. Now, it is obvious that, if the space where *the approximate field quantity* $\tilde{\boldsymbol{H}}$ "lives" is a suitable space for modeling the curl of $\tilde{\boldsymbol{H}}$, it cannot be, at the same time, suitable for modeling its divergence, as well.

Summarizing the above, however, it can be argued that we have a target space that is insufficiently wide (in fact, it is completely inadequate) for all required conditions to be accommodated.

In Jiang, Wu, and Povinelli (1996), it was suggested that the curl equations together with the divergence yield an overdetermination because there are more equations than unknowns. In our view, the target space provided for $\tilde{\boldsymbol{H}}$ may be insufficient (that is, of course, not to say that $\boldsymbol{H}$ was not properly discretized).

Boundary Simplicial Star. Next we study a simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$ that is related to a node located at a locally flat part of the outer boundary, *inside* a homogeneous subdomain of the domain of computation $\mathcal{D}$ and characterized by the constant permeability μ. Depending on the applicable boundary condition, either two or one degree(s) of freedom related to that node are (is) fixed by those boundary conditions (7) or (8), respectively. Consequently, only one or two equations remain for imposing the four conditions (1) and (2) that, like anywhere else, need to be imposed near the boundary. Just like in the situation of an internal simplicial star, we may again wonder if the employed variational formulation (17) provides the best possible solution for this combination of conditions to be satisfied. Is the target space sufficiently wide for the conditions to be accommodated?

Possible Effects of the Mesh. Once again we study a simplicial star $\mathcal{S}_{\mathcal{N}_{n(J)}}$ that is related to a node located at a locally flat part of the outer boundary, *inside* a homogeneous subdomain of the domain of computation $\mathcal{D}$ and characterized by the constant permeability μ. We now consider a two-dimensional detail of a specific mesh (see Figure 1) that may be found along the boundary. For simplicity, assuming that we have a boundary condition

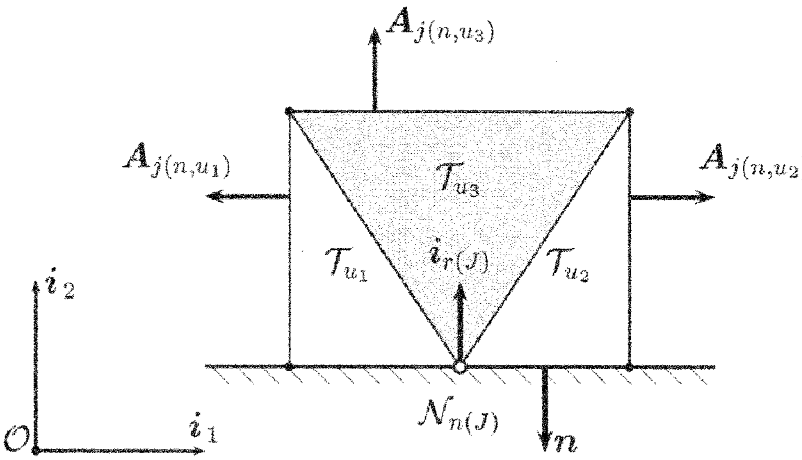

Figure 1. A detail of the mesh.

of type (7) and that we are using linear expansion functions, it easily follows that the application of (7) to this star causes the equations of the type (18) that are relevant to this star to reduce to *a single scalar equation*, for the only remaining unknown in this star. It is easily shown that this equation can be written as

$$\sum_{I\in\mathcal{S}_J} H_I \left[\left.\frac{(\boldsymbol{A}_j\times\boldsymbol{i}_{r_j})\cdot(\boldsymbol{A}_i\times\boldsymbol{i}_{r_i})}{9\,\mathrm{Vol}(\mathcal{T}_u)}\right|_{u=u_1} + \left.\frac{(\boldsymbol{A}_j\times\boldsymbol{i}_{r_j})\cdot(\boldsymbol{A}_i\times\boldsymbol{i}_{r_i})}{9\,\mathrm{Vol}(\mathcal{T}_t)}\right|_{u=u_2} \right.$$

$$\left. + \left.\frac{(\boldsymbol{A}_j\cdot\boldsymbol{i}_{r_j})\,(\boldsymbol{A}_i\cdot\boldsymbol{i}_{r_i})}{9\,\mathrm{Vol}(\mathcal{T}_u)}\right|_{u=u_3} \right] = \langle\text{source terms}\rangle. \qquad (19)$$

As regards the interpretation of this equation, we note that the terms in (18) containing curl ($\nabla\times$) operators have reduced to terms containing vector products and that the terms containing div ($\nabla\cdot$) operators have reduced to terms containing scalar products. Reading (19) in this way this equation cannot be interpreted as an attempt to variationally minimize the error in the curl and the divergence of the approximate solution of the field problem. It adds weighted values of the error in the curl in *some* tetrahedra ($\mathcal{T}_{u_1}$ and $\mathcal{T}_{u_2}$) to the weighted value of the error in the divergence in *another* tetrahedron ($\mathcal{T}_{u_3}$). It needs no further explanation that this equation therefore has no relevance to the minimization problem at hand and that it is in fact an erroneous equation.

Note 1. In the above text we have discussed a "two-dimensional detail" of the mesh using three-dimensional language. This is, of course, formally incorrect, but we do not doubt that the reader will nevertheless understand the point that is made.

Note 2. As regards the choice of the detail of the mesh, we note that it was merely taken to make our point as clearly as possible; other local details may be the cause of similar problems. The validity of a finite-element formulation should be independent of the mesh.

Note 3. It is not difficult to give a similar example for an internal simplicial star yielding the same type of erroneous equation.

On the Causes of Spurious Solutions

In this section we summarize our findings regarding the way in which (1)–(3) are modeled when using the local vectorial model. In the above section we found two different reasons for concern regarding the correctness of the local vectorial model:

1. Over the domain of computation $\mathcal{D}$ we are trying to satisfy four independent conditions (three for the curl and one for the divergence) while having only three independent degrees of freedom at our disposal. One can have doubts about a successful outcome of this, and more independent degrees of freedom (at least one extra) seem to be required.
2. The equations expressing our minimization procedure have been shown to " degenerate" near the outer boundary of the domain of computation where they, depending on the local mesh, may become explicitly erroneous. Degeneration of "internal" equations is also possible.

The difficulties mentioned above explain the spurious solutions that were observed when using an implementation of the local vectorial model. It needs to be mentioned here that no spurious solutions were observed for brick-shaped domains $\mathcal{D}$ filled with a homogeneous or a layered medium. As soon as we have a more complex domain of computation, e.g., one having a reentrant corner in its outer boundary, spurious solutions may be observed. An analysis of those spurious solutions has revealed that the largest local errors in the solution of (1)–(2) are found along the outer boundary of the domain of computation (i.e., at locations where the formulation degenerates) and near reentrant corners (where we have a singularity to boot).

Finally we note the following:

1. Instead of using generalized Cartesian expansion functions we could look for expansions that are free of divergence in the entire domain of computation (including at the interelement interfaces). Elements of that type would allow the use of a formulation accounting for the minimization of the error in the curl equation (1) only. For finite-element techniques, no expansions of that type seem to be available.
2. The degradation of the formulation shown has nothing to do with the complexity of the problem at hand. It may, and will, occur in even the simplest problems, and it is in fact surprising that so many correct results are obtained at all.

On Existing Methods that are Free of Spurious Solutions

Having concluded that we do have a problem, we need to start a quest for a better method. To that aim, we first notice that the world of computational electromagnetics has already produced a number of different methods that all have been proven to be free of spurious solutions (when applied correctly). We mention the following.

Integral equations. Integral equations *do allow* a solution in terms of the magnetic field strength $\boldsymbol{H}$ only, the reason being that they use Green's functions, which are exact solutions of the field equations to be solved. These functions exactly satisfy the divergence condition, consequently only the three conditions on the curl remain to be satisfied and three unknowns are available. For completeness we note that integral equation methods have shown spurious modes for eigenvalue problems as well as for driven problems including resonant cavities. Standard techniques are available here to avoid those spurious solutions.

Finite-differences in the time domain (FDTD). There is, of course, no static version of FDTD but it pays to have a look at the dimension of the target space FDTD uses. In FDTD both the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{H}$ serve as target space, and consequently there seems to be more room for all conditions to be satisfied.

Domain-integrated field equations. The domain-integrated field equation method is a finite-element–type method of computation. In it, both the field strengths and the fluxes are modeled using edge expansion functions for the fields and face expansion functions for the fluxes. The resulting method has a very wide target space that easily accommodates the conditions to be modeled.

On Methods that are Free of Spurious Solutions

Our analysis, together with the above observations, strongly suggests that a finite-element method for computing static magnetic fields cannot be based on the use of a target space consisting of the expansion of the magnetic field strength only. Most likely we have to add the magnetic flux density as an unknown vector function, at the same time removing the direct coupling (3) between $\boldsymbol{H}$ and $\boldsymbol{B}$. The method of solution we then have in mind is similar to the domain-integrated field equation method with the exception that it does, for reasons of improving its efficiency, not use edge and face elements but is based on the expansion of both $\boldsymbol{H}$ and $\boldsymbol{B}$ using generalized Cartesian expansion functions.

The formulation of the method then runs along the following lines:

1. Derive, for each elementary domain, a discretized version of (1).
2. Derive, for each elementary domain, a discretized version of (2).
3. Impose (3) in a weighted sense.

The resulting system of equations will be severely overdetermined and has to be solved by minimizing a suitably chosen norm.

A more detailed and general discussion of the subject discussed in the present paper, including more alternatives, is found in Lager and Mur (2000).

Conclusions

We have described some possible sources of spurious solutions in a class of finite-element methods for (electro)magnetic field problems. It was found that, to be able to model both the divergence and the curl accurately, it is not sufficient to include these conditions in the formulation. To eliminate spurious solutions it is necessary to provide a target space for the solution that is wide enough for accommodating both the divergence and the curl conditions in an appropriate manner. Failing to provide a wide enough target space causes spurious solutions that, at the level of the actual equations being implemented, are related to "degenerated" equations. To support this point of view, a number of available computational methods for electromagnetics were reviewed that are free of spurious solutions.

Finally, a sketch of a new finite-element method, free of spurious modes, was given.

References

Baldomir, D., & P. Hammond. 1996. *Geometry of electromagnetic systems*. Oxford, UK: Oxford University Press.

Bossavit, A. 1991. Differential forms and the computation of fields and forces in electromagnetism. *European Journal of Mechanics, B: Fluids*. 10(5):474–488.

Bossavit, A. 1998. *Computational electromagnetism, variational formulations, complementarity, edge elements*. San Diego: Academic Press.

Bossavit, A., & L. Kettunen. 1999. Yee-like schemes on staggered cellular grids: A synthesis between FIT and FEM approaches. In *COMPUMAG*. *12th Conference on the Comuptation of Electromagentic Fields*, Sapporo, pp. 2–3.

Choquet-Bruhat, Y., & C. DeWitt-Morette. 1982. *Analysis, manifolds and physics*. Amsterdam: North-Holland.

de Hoop, A. T., & I. E. Lager. 1998. Static magnetic field computation—An approach based on the domain-integrated field equations. *IEEE Transactions on Magnetics* 34(5):3355–3358.

de Hoop, A. T., & I. E. Lager. 1999. The domain-integrated software package for computing quasi-static magnetic fields. *Software for Electrical Engineering. Analysis and Design IV*, ed. A. Konrad and C. A. Brebbia, pp. 223–232. Southamptom, UK: WITPRESS.

Jiang, B., J. Wu, & L. A. Povinelli. 1996. The origin of spurious solutions in computational electromagnetics. *Journal of Computational Physics* 125:104–123.

Lager, I. E., & G. Mur. 1998a. Generalized Cartesian finite elements. *IEEE Transactions on Magnetics* 34(4):2220–2227.

Lager, I. E., & G. Mur. 1998b. Least-squares minimising finite-element formulation for static and stationary electric and magnetic fields. *IEEE Transactions on Magnetics* 34(5):2419–2424.

Lager, I. E., & G. Mur. 2000. *Finite element methods for electromagnetics*. Scientific report IRCTR-S-003-00; also Internal report ET/Et 2000-07, available from the authors.

Mur, G. 1994a. Compatibility relations and the finite-element formulation of electromagnetic field problems. *IEEE Transactions on Magnetics* 30(5):2972–2975.

Mur, G. 1994b. Edge elements, their advantages and their disadvantages. *IEEE Transactions on Magnetics* 30(5):3552–3557.

Mur, G. 1998. The fallacy of edge elements. *IEEE Transactions on Magnetics* 34(5):3244–3247.