A finite–element method for the modeling of electromagnetic fields using generalized Cartesian elements

Gerrit Mur
IRCTR, Faculty of Informationtechnology and Systems, Delft University of Technology

Ioan E. Lager
Laboratory of Electromagnetic Research, Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2628 CD Delft, The Netherlands

Abstract– **This paper presents a new finite–element method for the modeling of three–dimensional electromagnetic fields in inhomogeneous media. For the accurate and efficient modeling of the field along interfaces between media with different constitutive parameters, the family of generalized Cartesian finite elements is applied, that replaces the commonly used edge elements. A simple numerical example demonstrating the use of generalized Cartesian finite–elements, when applied to stationary magnetic fields, is presented.**

Index terms– **Finite–elements, generalized Cartesian elements.**

I. Introduction

The design of a finite element method for computing electromagnetic fields in inhomogeneous media is, when compared with other applications of finite–elements such as structural engineering, complicated by the fact that electromagnetic fields show discontinuities across interfaces between different media. Traditionally, difficulties of this type forced the designer to make a choice: either (s)he uses a (vector) potential–based finite–element method in which case the formulation can be such that all components of the (vector) potentials are continuous [1], or (s)he chooses a finite–element method that takes these discontinuities into account by using edge/face(t) elements. Both solutions have the disadvantage of being inefficient. Potentials are inefficient because of requiring a numerical differentiation of the vector potential for calculating the electric and/or magnetic field quantities that have physical meaning and, hence, are of interest in engineering applications [2]. Representing a field by using edge/face elements yields inefficient bases and the polynomials used often are incomplete [3], [4].

Recently, generalized Cartesian elements were introduced that can, like edge elements, be used for representing fields having discontinuities across interfaces [5]. Contrary to edge elements, these elements yield an efficient and complete representation of the field in a manner that, in interface–free subdomains, is identical to the representation of the classical nodal elements. Along interfaces between regions containing different media, generalized Cartesian elements yield a representation of the field that exactly models the local continuity conditions applying to both the tangential components of the field strength and the normal component of the flux density. Using these elements, one obtains an efficient representation of the field distribution together with the automatic satisfaction of all continuity conditions along locally smooth interfaces between different media.

Generalized Cartesian elements cannot be employed at points where the interface between different media, or the outer boundary, is not locally smooth [5]. In fact, at such points the field will in general be singular [6] and cannot be represented accurately by using first–order polynomials. The discussion of this situation is outside the scope of the present paper. For the time being, several practical solutions are considered for representing field quantities at such points: (generalized) Cartesian expansion functions associated with incomplete simplicial stars (as suggested in [5]), completely linear edge elements [7] (solution employed in this paper), etc.

II. Generalized Cartesian elements

The difficulty of representing the field along interfaces can be solved by using expansion functions that are not merely (combinations of) simple mathematical functions (usually polynomials) over simple, simply connected domains, e.g. simplices (tetrahedra in three–dimensional configurations), but that include the relative contrast in the constitutive parameters across the interface as well. In other words by using expansion functions that do not depend only on geometric data of the mesh but that may also depend on a relative change in the material properties. We shall refer to this type of expansion functions as *generalized Cartesian expansion functions*, generalized because of the inclusion of material data and Cartesian because we will represent the Cartesian components of the field. Note that with the choice for the name Cartesian (which replaces nodal) we have a naming of our elements that is in accordance with the bases used (as is the case with edge and face elements). Topologically, all elements (wether of the edge, face or Cartesian type) are related to the node defining the span of those elements. The different bases used by them are characteristic properties that

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G. Mur, 31-15-2786294, fax 31-15-2786194, g.mur@its.tudelft.nl, http://embib.et.tudelft.nl/index.html; I. E. Lager, 31-15-2784429, i.lager@its.tudelft.nl.

indicate the type (quality) of the representation of the vector field to be computed.

In Figure 1, a one–dimensional example is given for a first–order generalized expansion function. In Figure 1a we see the well–known first–order, continuous expansion consisting of two linear functions of the horizontal spatial coordinate. The use of this type of expansions will yield a continuous and piecewise linear representation of the expanded function. In Figure 1b we see a first–order, generalized expansion again consisting of two linear functions. Note that the expansion function can be chosen such that it accurately models any given discontinuity due to a relative contrast α in the unknown function to be represented. Consequently, functions of this type can be used to represent piecewise linear functions including known discontinuities due to discontinuities in the constitutive parameters.

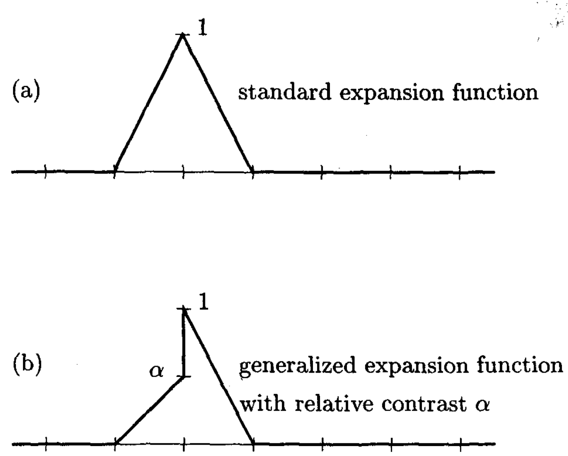

Fig. 1. Standard (a) and generalized (b) expansion functions (1–D).

Obviously, the one–dimensional scalar example of Figure 1 allows a straightforward generalization to three–dimensional expansion functions for the Cartesian components of a vector field. We shall refer to these functions as generalized Cartesian expansion functions. For expanding, for instance, the three Cartesian components of an electric field along a locally flat interface, a generalized Cartesian (vectorial) expansion function will be employed, of which the two components that are parallel to this interface are identical to the functions used in an interface–free subdomain, whereas the normal component is chosen to satisfy the local (dis)continuity condition. Finally, we note that the linear generalized Cartesian elements presented above can easily be generalized to elements of higher order polynomial degrees.

III. The field equations

Let $\mathcal{D}$ be an open domain of computation in $\mathbb{R}^3$ with outer boundary $\partial\mathcal{D}$. A finite number of interfaces $\mathcal{I}$ between media with different constitutive parameters can be present inside $\mathcal{D}$. The unit vectors along the outward normal on $\partial\mathcal{D}$ and on interfaces are denoted as $\boldsymbol{n}$ and $\boldsymbol{\nu}$, respectively. The media inside the domain of computation are linear. For reasons of simplicity, only, the materials inside $\mathcal{D}$ are taken to be isotropic. The following equations hold in the static and stationary limit of magnetic fields [8]

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{\mathrm{imp}} \quad \text{in } \mathcal{D}\backslash\mathcal{I}, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{in } \mathcal{D}\backslash\mathcal{I}, \tag{2}$$

$$\boldsymbol{B} = \mu\, \boldsymbol{H}, \tag{3}$$

$$\boldsymbol{\nu} \times \boldsymbol{H}\big|_1^2 = \boldsymbol{J}_{\mathrm{S}}^{\mathrm{imp}} \quad \text{across } \mathcal{I}, \tag{4}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{B}\big|_1^2 = 0 \quad \text{across } \mathcal{I}, \tag{5}$$

$$\boldsymbol{n} \times \boldsymbol{H} = \boldsymbol{n} \times \boldsymbol{H}^{\mathrm{ext}} \quad \text{on } \partial\mathcal{D}_H, \tag{6}$$

$$\boldsymbol{n} \cdot \boldsymbol{B} = -\boldsymbol{n} \cdot \boldsymbol{B}^{\mathrm{ext}} \quad \text{on } \partial\mathcal{D}_B, \tag{7}$$

where $\boldsymbol{H}$ denotes the magnetic field strength, $\boldsymbol{B}$ the magnetic flux density, μ the permeability, $\boldsymbol{J}^{\mathrm{imp}}$ the source density of impressed electric current, $\boldsymbol{J}_{\mathrm{S}}^{\mathrm{imp}}$ the source density of impressed electric surface current ($\boldsymbol{J}_{\mathrm{S}}^{\mathrm{imp}}$ satisfies the condition $\boldsymbol{\nu} \cdot \boldsymbol{J}_{\mathrm{S}}^{\mathrm{imp}} = 0$). The outer boundary is partitioned in two sub–surfaces $\partial\mathcal{D}_H$ and $\partial\mathcal{D}_B$ such that $\partial\mathcal{D}_H \cup \partial\mathcal{D}_B = \partial\mathcal{D}$ and $\partial\mathcal{D}_H \cap \partial\mathcal{D}_B = \emptyset$. $\partial\mathcal{D}_H$ is always connected. $\boldsymbol{n} \times \boldsymbol{H}^{\mathrm{ext}}$ and $\boldsymbol{n} \cdot \boldsymbol{B}^{\mathrm{ext}}$ denote known vectorial and scalar functions, respectively, that are used for prescribing boundary conditions. Equations (1)–(7) define a problem with a unique solution [8].

IV. The formulation

In [9], [10] it was concluded that reliable computational results from finite–element methods for solving the electromagnetic field equations can only be obtained by making both the electromagnetic field equations and the appertaining electromagnetic compatibility relations (divergence conditions, interface conditions and outer boundary conditions) a part of the formulation of the problem. In our approach we satisfy these conditions in the following manner:

1. When using generalized Cartesian elements, the interface conditions are imposed exactly along all locally flat interfaces. Outer boundary conditions can be imposed exactly using standard Cartesian expansion functions along all locally flat parts of the outer boundary.

2. At points where the interface and the outer boundary conditions cannot be imposed (when boundaries are not locally flat) standard elements must be used, in a manner that allows obtaining a reasonably accurate solution. Obviously, the best choice would be to use elements that have the proper degree of singularity at these locations but the way to implement this remains a matter of further research. For a detailed analysis of situations near singularities with respect to the number of degrees of freedom available and the freedom to impose continuity conditions the reader is referred to [5].

3. Finally, away from the outer boundary and the interfaces, the curl and the divergence condition are made a part of the formulation by using a standard Galerkin weighting procedure such that the error in both the curl and the divergence are simultaneously minimized. By using Galerkin's method we obtain a symmetric system of linear algebraic equations that can, for simple cases, be shown to solve the field problem in the least–squares sense [11].

V. A NUMERICAL EXAMPLE

For demonstrating the validity of our approach, we present results for a very simple cylindrical test configuration as depicted in Figure 2. All dimensions in the x, y–plane of this figure are integer multiples of 1m. The copper conductor carries a uniform current distribution with a total direct current of 100A. Although the configuration is cylindrical in the z–direction, and therefore essentially two–dimensional, all computations were carried out in three dimensions using a finite–element mesh with only one layer of tetrahedra in the z–direction. As regards the boundary conditions, we note that the normal component of the magnetic density flux is set to zero ($\boldsymbol{n} \cdot \boldsymbol{B}^{\text{ext}} = 0$) at the plane $x = 0$ and at the planes bounding the configuration in the z–direction, while at the remaining parts of the outer boundary, the tangential components of the magnetic field strength are set to zero ($\boldsymbol{n} \times \boldsymbol{H}^{\text{ext}} = \boldsymbol{0}$).

For a first validation of our approach, a computational experiment was carried out on a modified version of the FEMAXS finite-element package [12]. This package was modified such that we could use linear generalized Cartesian elements along all locally flat parts of the interfaces. Generalized Cartesian could not be employed along the edges of the interfaces ($x = 1$, $y = 1$) and ($x = 1$, $y = 3$), where was made of consistently linear edge elements at the nodes located along those edges.

As regards the mesh, we note that it has a mesh–size of 0.1m near the locations where the field is expected to be singular (i.e. near the edges of the interface) and an increasing mesh–size away from these locations. Numerical results for $H_x(x, y)$ and $H_y(x, y)$ are given in Fig. 3. Note the excellent (in fact exact) satisfaction of the interface conditions at the plane $x = 1$, for $H_x(x, y)$ and at the planes $y = 1$ and $y = 3$, for $H_y(x, y)$. By computing the line integral of the projection of $\boldsymbol{H}$ on relevant closed paths inside the domain of computation, we know that the error in modelling Ampère's law is in the order of 1%.

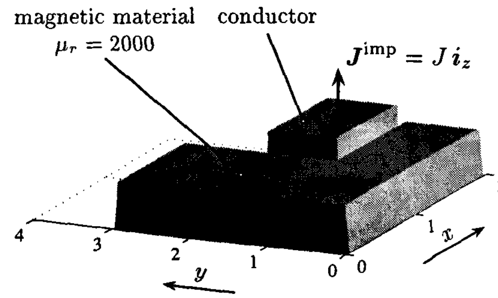

Fig. 2. The configuration for the "Test slot" problem; the modulus of the impressed density of current is $J = 100\,\text{A/m}^2$.

We are of the opinion that, although this example does not exactly represent the method we have in mind (a few edge elements were used along the edges of the interfaces), the results clearly demonstrate the validity and the accuracy of our approach and the fact that (large) contrasts can be modeled exactly.

VI. CONCLUSIONS

We have presented a finite–element method for modeling three–dimensional electromagnetic fields in inhomogeneous media. Generalized Cartesian expansion functions, that exactly model all local continuity conditions applying to the electromagnetic field quantities, were employed. In this manner, extremely accurate numerical solutions for electromagnetic field problems are made possible, without unnecessarily increasing the computational costs, even in the case of high contrasts in the parameters of the media inside the analysed configuration.

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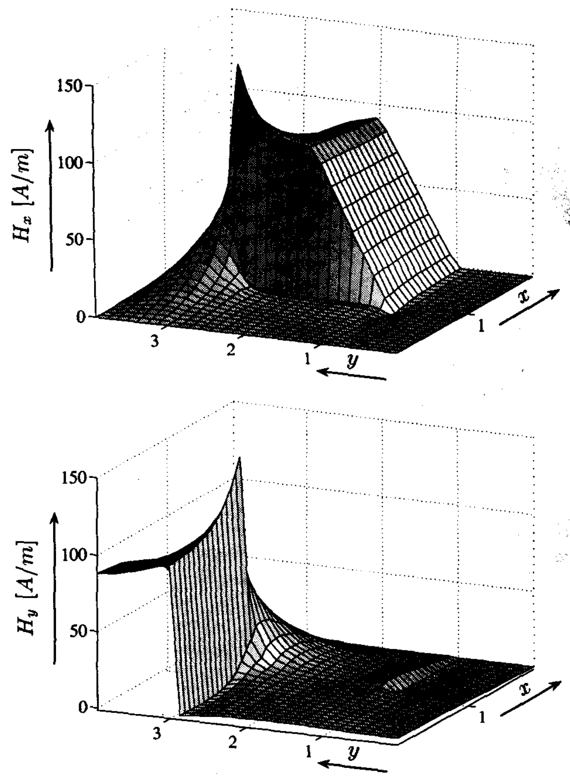

Fig. 3. The magnetic field strength in the "Test slot" problem.

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