Application of the Domain-Integrated Field Relations Method to the Solution of Large Scale Static and Stationary Magnetic Field Problems

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***Abstract*—A new efficient implementation of the domain-integrated field relations approach to computing static and stationary magnetic fields is presented. The philosophy behind this implementation is to combine the robustness of this method with a computationally efficient discretization technique that uses first-order Cartesian expansion functions on a simplicial mesh. The accuracy of the implementation is illustrated by solving a two-dimensional field problem. The ability of the method to accurately model the behavior of the field at interfaces between highly contrasting media and at geometrical singularities is emphasized. A detailed efficiency analysis of the present implementation is performed.**

***Index Terms*—Magnetic fields, numerical analysis.**

I. Introduction

THE DOMAIN-integrated field relations approach to computing static and stationary magnetic fields (see [1] and references therein) has been extensively proven to represent a robust and accurate method for modeling the field in highly inhomogeneous anisotropic configurations. No erratic solutions have been noticed in any of the numerous numerical experiments carried out thus far in relation to this computational method. The method requires the expansion of *two* field quantities (i.e., the magnetic field strength $\boldsymbol{H}$ as well as the magnetic flux density $\boldsymbol{B}$) and employs an expansion technique based on the use of consistently linear edge and face expansion functions [2]. However, these expansion functions are known to be computationally (very) inefficient. Consequently, the feasibility of large scale applications of the method, needed for solving field problems in configurations of practical interest, is questionable. For tackling practical applications, a *drastic* improvement of the efficiency of the method is stringently required.

It is first noted that the expansion of *one single* field quantity can easily be ruled out (since it conflicts with the difference in the physical character of $\boldsymbol{H}$ and $\boldsymbol{B}$, reflected in, for instance, the different behavior of these quantities at material interfaces). The only option for increasing the efficiency of the method is then identifying an alternative to the use of edge and face expansion functions. In this respect, the use of the computationally efficient Cartesian expansion offers the appropriate answer. Additionally, the use of this expansion technique also provides a remedy to the "roughness" of the modeling of field quantities by means of edge and face expansion functions, that clearly conflicts with these quantities being *continuously differentiable* functions of the spatial coordinates inside interface-free subdomains.

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II. Computational Approach

Let $\mathcal{D} \subset \mathbb{R}^3$ be a bounded domain of computation with a piecewise smooth outer boundary $\partial\mathcal{D}$. The magnetic field inside $\mathcal{D}$ is excited by means of a piecewise continuous volume density of electric current $\boldsymbol{J}$ (with bounded support). Inside $\mathcal{D}$, the positive definite tensorial permeability of rank two $\boldsymbol{\mu} = \boldsymbol{\mu}_r \cdot \mu_0$ (with $\mu_0 = 4\pi \cdot 10^{-7}$ H/m) varies piecewise continuously with position. It is assumed that the media behave linearly. On the boundary $\partial\mathcal{D}$, explicit boundary conditions are prescribed such that the thus defined field problem has a unique solution.

The domain of computation is decomposed into mutually disjoint open simplicial cells (tetrahedra in $\mathbb{R}^3$ and triangles in $\mathbb{R}^2$) $\mathcal{T}_m$ $(m = 1 \cdots M)$. (Parts of) material interfaces and of the outer boundary $\partial\mathcal{D}$ may only coincide with the boundaries $\partial\mathcal{T}_m$ of the simplicial cells $\mathcal{T}_m$. The outward normal to $\partial\mathcal{T}_m$ is denoted as $\boldsymbol{n}$. The field quantities $\mathbf{H}$ and $\boldsymbol{B}' = \boldsymbol{B}/\mu_0$ are expanded separately in terms of first-order vector expansion functions. For being able to appropriately account for the continuity conditions inside interface-free subdomains and at material interfaces, a special type of expansion is employed. This expansion technique is hereafter discussed in detail.

A. Expansion Technique

The first (extreme) case in our analysis is that of *strongly inhomogeneous configurations.* In this situation there does not seem to exist an alternative to the expansion technique described in [1] (the one based on the use of consistently linear edge and face expansion functions). Nevertheless, most configurations that occur in practical applications consist of a relatively reduced collection of subdomains with constitutive material parameters that vary continuously in space (hereafter referred to as interface-free subdomains). It then follows that, for (practically all) technical configurations, this case can be ignored.

The diametrically opposed (extreme) situation is that of *interface-free subdomains*, which occupy, in fact, the most extended part of the vast majority of technical configurations. Inside such

subdomains the field quantities are continuously differentiable functions. It then follows that in this case, employing standard Cartesian expansion functions, which ensure the continuity of expanded quantities across common faces of adjacent simplicial cells, is possible. Apart from resulting in a significant reduction of the number of the degrees of freedom (DOFs) to be computed, this expansion technique allows a smooth representation of the field quantities, which should be the case inside interface-free subdomains.

At (locally) flat material interfaces, the standard interface boundary conditions that apply to static and stationary magnetic fields require the employed expansion to ensure the continuity (in machine accuracy) of the *tangential* components of $\boldsymbol{H}$ and of the *normal* component of $\boldsymbol{B}'$, while leaving the complementary component(s) free to jump. For accommodating these conditions, an expansion technique that is reminiscent of the method employed for deriving the generalized Cartesian expansion functions [3] is used. Practically, given $\mathcal{N}_n$ is a node located at a (locally) flat interface, its simplicial star $\mathcal{S}_n$ is split into two parts $\mathcal{S}_n'$ and $\mathcal{S}_n''$, located on one side and the other of the interface. Subsequently, DOFs associated with $\mathcal{S}_n$ are allocated to the *continuous* components of the field quantities, while *doublets* of DOFs, associated with $\mathcal{S}_n'$ and $\mathcal{S}_n''$, respectively, are allocated to the *discontinuous* ones. This field representation accounts for *all continuity requirements* inside $\mathcal{S}_n$ and introduces the minimum roughness that is needed for coping with the field behavior at the interface. This choice for field expansion results in a reduction of the number of DOFs to be computed as well. Finally, unlike in the case of generalized Cartesian expansion functions, the expansion technique described above makes no reference to *explicit values* of the contrast in the material parameters on the two sides of the interface. In this manner, the implementation of the expansion technique in a computer code is largely simplified, especially in the case when the interfaced media are anisotropic or nonlinear.

The last and most difficult case is that of *geometrical singularities.* Let $\mathcal{N}_n$ be a node located at a geometrical singularity and let $\mathcal{S}_n$ be its simplicial star. For ensuring the consistency of our expansion technique, a Cartesian expansion should be employed in this case as well. To this end, for endowing our expansion technique with the freedom required for coping with the singular behavior of the field in the vicinity of $\mathcal{N}_n$, the Cartesian expansion functions relating to this node are associated with each simplicial cell $\mathcal{T}_m \subset \mathcal{S}_n$ taken separately. In this manner, for each simplicial cell $\mathcal{T}_m \subset \mathcal{S}_n$, two *triplets* of DOFs (one for $\boldsymbol{H}$ and one for $\boldsymbol{B}'$) associated with the node $\mathcal{N}_n$ are allocated. This type of expansion is reminiscent of the use of consistently linear edge and face expansion functions, with the provision that no continuity is enforced anymore.

The abrupt change of the representation at common faces of adjacent simplicial cells, of those components that are to be continuous as well, results apparently into the generation of unphysical surface sources, which may have a detrimental effect on the accuracy of the computed solution. Nevertheless, the domain-integrated relations method is based on enforcing the accuracy of the integrals of $\boldsymbol{n} \times \boldsymbol{H}$ and $\boldsymbol{n} \cdot \boldsymbol{B}'$ on the boundaries of "elementary" domains. Now, since the field is singular "at" $\mathcal{N}_n$, this node must be isolated by means of a closed surface of vanishing maximum distance (in our computational scheme, we take this to be the boundary $\partial\mathcal{S}_n$ of $\mathcal{S}_n$). It can be shown that when the continuity requirements are satisfied along $\partial\mathcal{S}_n$, which is the case since $\partial\mathcal{S}_n$ either is located inside interface-free subdomains or intersects (locally) flat interfaces, imposing Ampère's/Gauss' laws on the boundaries $\partial\mathcal{T}_m$ of each simplicial cell $\mathcal{T}_m \subset \mathcal{S}_n$ results in imposing these laws on $\partial\mathcal{S}_n$ at least with $O(h)$ accuracy. (This accuracy becomes $O(h^2)$ when no interfaces are intersected by $\partial\mathcal{S}_n$.) It can now be concluded that the expansion strategy employed for modeling singularities fully complies with the assumptions on which the domain-integrated method was developed.

B. Numerical Formalism

The discretized field quantities are now substituted in the domain-integrated field relations [1]. In view of ensuring their *physical homogeneity,* appropriate scaling factors are employed. Summarizing, the present implementation of the domain-integrated field relations method employs for each simplicial cell $\mathcal{T}_m$ $(m = 1 \ldots M)$ the following (the required scaling factors being explicitly specified for each equation):

$$\int_{\partial\mathcal{T}_m} \boldsymbol{n} \times \boldsymbol{H}\,\mathrm{d}A = \int_{\mathcal{T}_m} \mathbf{J}\,\mathrm{d}V, \quad \left|\left(h^{-2}\|\tilde{\boldsymbol{\mu}}_r\|_{\mathcal{T}_m}^{1/2}\right)\right. \tag{1}$$

$$\int_{\partial\mathcal{T}_m} \boldsymbol{n} \cdot \boldsymbol{B}'\mathrm{d}A = 0, \quad \left|\left(h^{-2}\|\tilde{\boldsymbol{\mu}}_r\|_{\mathcal{T}_m}^{-1/2}\right)\right. \tag{2}$$

$$\int_{\mathcal{T}_m} \boldsymbol{W}_J^{\boldsymbol{B}} \cdot (\boldsymbol{B}' - \tilde{\boldsymbol{\mu}}_r \cdot \boldsymbol{H})\,\mathrm{d}V = 0, \quad \left|\left(h^{-3}\|\tilde{\boldsymbol{\mu}}_r\|_{\mathcal{T}_m}^{-1/2}\right)\right. \quad \text{for } \forall\, \boldsymbol{W}_J^{\boldsymbol{B}} \in \mathcal{T}_m \tag{3}$$

where h is the maximum length of an edge pertaining to $\mathcal{T}_m$, $\tilde{\boldsymbol{\mu}}_r$ denotes an "effective" relative permeability in $\mathcal{T}_m$ ($\tilde{\boldsymbol{\mu}}_r$ being taken to be constant in $\mathcal{T}_m$), $\|\tilde{\boldsymbol{\mu}}_r\|_{\mathcal{T}_m} = \|\tilde{\boldsymbol{\mu}}_r\|_\infty$ in $\mathcal{T}_m$, and $\boldsymbol{W}_J^{\boldsymbol{B}} \in \mathcal{T}_m$ are the expansion functions that expand $\boldsymbol{B}'$ in $\mathcal{T}_m$. If some of the expansions of $\boldsymbol{B}'$ in $\mathcal{T}_m$ are prescribed by means of boundary conditions, the relevant equations of type (3) are replaced by equations of the type

$$\int_{\mathcal{T}_m} \boldsymbol{W}_J^{\boldsymbol{H}} \cdot (\boldsymbol{B}' \cdot \tilde{\boldsymbol{\mu}}_r^{-1} - \boldsymbol{H})\,\mathrm{d}V = 0, \quad \left|\left(h^{-3}\|\tilde{\boldsymbol{\mu}}_r\|_{\mathcal{T}_m}^{1/2}\right)\right. \quad \text{for } \boldsymbol{W}_J^{\boldsymbol{H}} \in \mathcal{T}_m \tag{4}$$

where $\boldsymbol{W}_J^{\boldsymbol{H}} \in \mathcal{T}_m$ are the expansion functions that expand $\boldsymbol{H}$ in $\mathcal{T}_m$. For solving the thus resulting overdetermined system of linear algebraic equations, the system's matrix needs to be multiplied by its transpose. The relevant product is evaluated efficiently by keeping track of the band structure of the matrix and carrying out all operations at element level. In this manner, the construction of the final form of the system's matrix amounts to summing the relevant element contributions.

III. Numerical Example

For validating our theoretical ideas, a two-dimensional (2-D) version of the method was implemented in a C++ package called Des_Cartes. In the present contribution, the accuracy and the robustness of this implementation is demonstrated by solving a

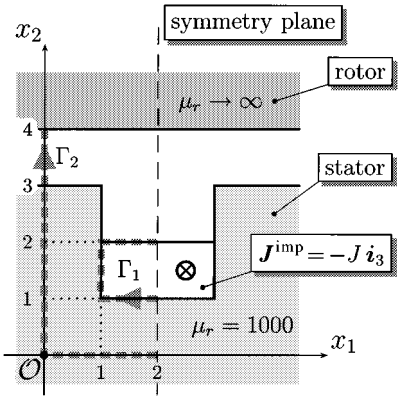

Fig. 1. Configuration concerning the test-slot problem $J = 1\,\text{A} \cdot \text{m}^{-2}$. Impressed boundary conditions $\boldsymbol{n} \times \boldsymbol{H} = 0$ along $x_1 = 2$ and $x_2 = 4$ and $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ otherwise. Dimensions are in meters.

TABLE I
LINE INTEGRALS FOR THE TEST-SLOT PROBLEM

Rectangular mesh density	# of DoFs	$\int_{\Gamma_1} \boldsymbol{\tau} \cdot \boldsymbol{H}\, \mathrm{d}s$ (A/m)	$\int_{\Gamma_2} \boldsymbol{\tau} \cdot \boldsymbol{H}\, \mathrm{d}s$ (A/m)
4×8	382	0.9402	0.7672
8×16	1218	0.9714	0.8892
16×32	4426	0.9850	0.9417
32×64	16986	0.9891	0.9602
64×128	66682	0.9882	0.9610

field problem concerning a realistic configuration in the realm of electrical machines, depicted in Fig. 1 (a problem that is hereafter referred to as the test-slot problem). The domain of computation was discretized by means of an increasingly fine *uniform mesh* (see Table I) consisting of quadrilateral cells, each of which being subdivided into four triangles, delimited by the cells' diagonals. The resulting system of linear, algebraic equations was solved using a preconditioned conjugate gradient iterative solver. Since the conditioning of the system of equations provided by our method is very good, employing an elementary (diagonal scaling) preconditioner turns out to be sufficient for solving it. The accuracy in modeling the field equations is demonstrated by computing the line integral of $\boldsymbol{\tau} \cdot \boldsymbol{H}$ along the curves Γ_1 and Γ_2, indicated in Fig. 1. The results of this analysis are summarized in Table I. Note that the accuracy of the computed results is maintained even for very large numbers of DOFs. The ability of our method to cope with the singularities of the field at the corners of the highly permeable slot (located at $\{x_1, x_2\} = \{1, 1\}$ and $\{1, 3\}$, respectively) is demonstrated by Fig. 2. In these plots, the singular behavior of the field near the edges of the slot is clearly visible.

IV. COMPARISON WITH THE PREVIOUS VERSION OF THE METHOD

A 2-D version of the domain-integrated field relation method, as described in [1] (employing consistently linear edge and face expansion functions), was implemented in a C++ package called domain_integrated. It should be mentioned that the domain_integrated package employs the new scaling procedure indicated in Section II. This choice resulted in a noticeable improvement

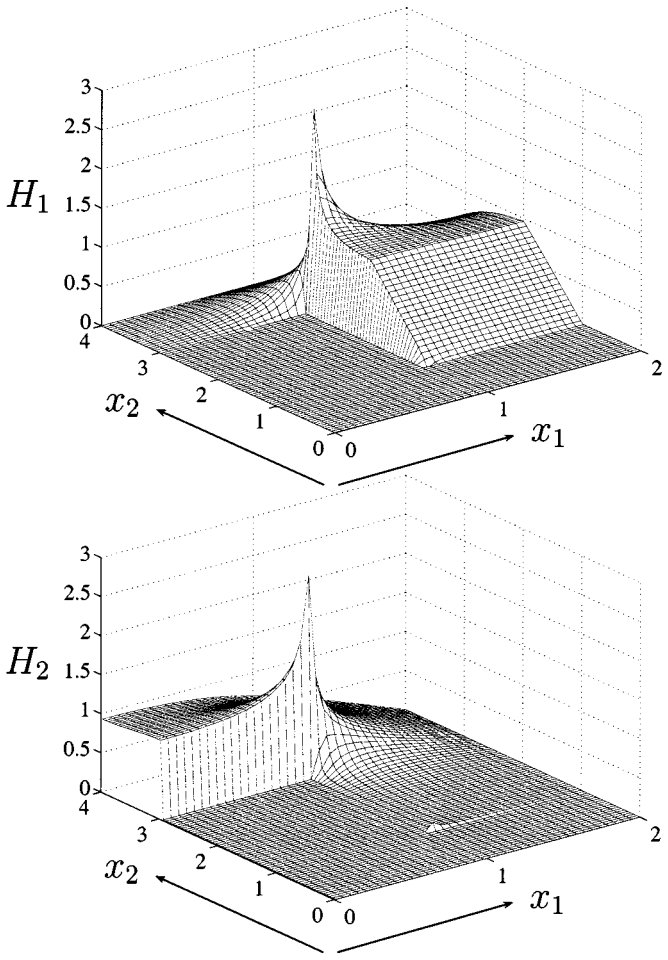

Fig. 2. Distribution of the magnetic field strength for the test-slot problem. Mesh consists of 64×128 rectangular cells.

of the method's computational stability when compared to the implementation reported in [1].

For illustrating the further increase of the computational stability resulting from the use of the expansion technique based on the Cartesian expansion, a comparison between computational results obtained by means of the domain_integrated and Des_Cartes packages is provided. It must be mentioned that both packages employ for the geometrical discretization a uniform division of the domain of computation into rectangular cells, followed by a division of each cell into four triangles, delimited by the cells' diagonals. It then follows that when the initial cells are quadrilateral, the representation basis for the edge and face expansion functions are orthogonal. Consequently, the accuracy of the representation of vector quantities by means of these basis is comparable with that provided by the background—Cartesian base. However, when the initial cells are elongated, the use of edge and face expansion functions suffers from the well-known deterioration of the accuracy of the representation of vector quantities in very flat basis, while the formulation using Cartesian expansion functions is less sensitive to the aspect ratio of the mesh. This effect will be clearly reflected by our computational results.

The first example concerns the solution of the test-slot problem, as defined in Section II. The meshes employed by

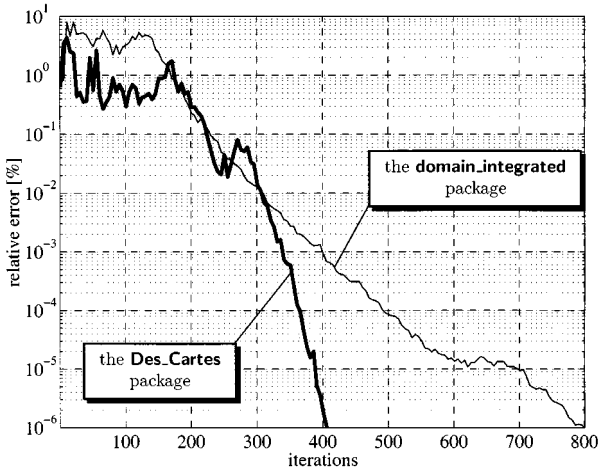

Fig. 3. Convergence of the iterative process for the test-slot problem (solver: CG with diagonal scaling preconditioner).

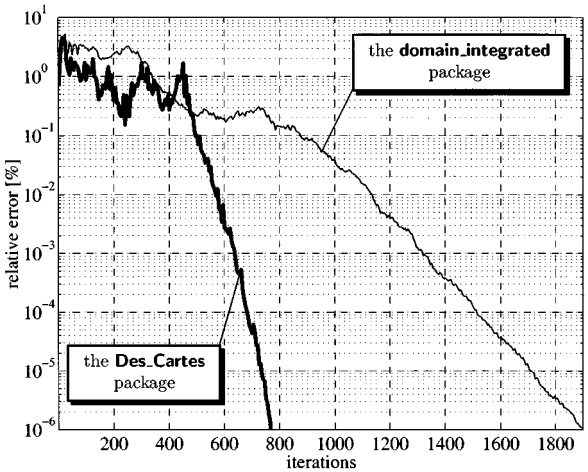

Fig. 4. Convergence of the iterative process for the flattened test-slot problem (solver: CG with diagonal scaling preconditioner).

the domain_integrated and Des_Cartes packages, respectively, were chosen such that the corresponding numbers of DOFs were approximately the same. The system of linear equations was solved using a conjugate gradient method in conjunction with a diagonal scaling preconditioner. The required accuracy of the solution of the system was 10^{-6}. Denoting χ as the condition number, A^{raw} as the system's matrix before preconditioning, and A^{prec} as the preconditioned matrix, the basic computational data referring to this problem were as follows.

1) domain_integrated package:
   a) mesh density: 16×32 quadrilateral cells;
   b) number of DOFs: 12 480;
   c) $\chi(A^{\text{raw}}) = 8.40 \cdot 10^{10}$;
   d) $\chi(A^{\text{prec}}) = 2.46 \cdot 10^{8}$.
2) Des_Cartes package:
   a) mesh density: 28×56 quadrilateral cells;
   b) number of DOFs: 12 876;
   c) $\chi(A^{\text{raw}}) = 4.48 \cdot 10^{11}$;
   d) $\chi(A^{\text{prec}}) = 3.36 \cdot 10^{6}$.

Additionally, the convergence of the iterative process in the case of the two packages is depicted in Fig. 3. Based on these facts, it can be concluded that Des_Cartes provides a clearly superior performance: for the same mesh density (hence, for a comparable accuracy of the expansion), it requires significantly less DOFs, while the iterative process for comparable numbers of DOFs is considerably faster. Note that the choice for the diagonal scaling preconditioner is dictated by the memory requirements that, for large-scale problems, become of paramount importance.

Second, for analyzing the effect of the deterioration of the aspect ratio of the mesh on the effectiveness of the method, the solution of a variant of the test-slot problem (hereafter referred to as the flattened test-slot problem) is presented. In this case, the width of the slot (see Fig. 1) was reduced by a factor of five, while the vertical dimensions were kept the same. The employed meshes were similar to the ones employed in the case of the test-slot problem, the only difference being that, in this case, the rectangular cells had a height/width ratio of five. The iterative solver's parameters were the same as those employed for the test-slot problem. The condition numbers referring to the flattened test-slot problem were as follows.

1) domain_integrated package:
   a) $\chi(A^{\text{raw}}) = 4.97 \cdot 10^{11}$;
   b) $\chi(A^{\text{prec}}) = 3.49 \cdot 10^{7}$.
2) Des_Cartes package:
   a) $\chi(A^{\text{raw}}) = 2.30 \cdot 10^{12}$;
   b) $\chi(A^{\text{prec}}) = 7.77 \cdot 10^{6}$.

Additionally, the convergence of the iterative process in the case of the two packages is depicted in Fig. 4. Apart from confirming the conclusions drawn for a quadrilateral mesh, these data indicate that domain_integrated is more sensitive to the aspect ratio of the mesh, since the total number of iterations increased by a factor of $\approx$2.4 with respect to the quadrilateral mesh, while for Des_Cartes, this factor was $\approx$1.9, only.

V. Conclusion

A new efficient implementation of the domain-integrated field relations approach to computing static and stationary magnetic fields was presented. The computational model is both accurate and robust. It models accurately the behavior of the field at interfaces between highly contrasting media and in the vicinity of points where the field is singular. The conditioning of the system of linear algebraic equations that follows from its application is very good. The computational model lends itself to the solution of large-scale field problems.

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